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Rationality of the zeta function for Ruelle-expanding maps

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Abstract

We will prove that the zeta function for *Ruelle-expanding* maps is rational.

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1 Introduction

For any map f with a finite number of periodic points for each period we can associate its *zeta function* defined as

$$\zeta_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n(f)}{n} t^n\right)$$

where $N_n(f)$ is the number of periodic points with period n. In some cases, it is known that $\zeta_f(t)$ is a rational function. Those cases include the Markov subshifts of finite type (unilateral and bilateral) and Axiom A diffeomorphisms. Besides, in the case of the subshifts, an explicit formula relates the topological entropy and the radius of convergence of the zeta function. Another class of maps with this property is the *Ruelle-expanding* maps. This concept, created by Ruelle, generalizes the notion of expanding maps defined on manifolds, freeing its essence from the derivative's constraints. Our main result will be the following

Theorem 1.1 *If f is Ruelle-expanding, then its zeta function is rational.*

Its proof will emulate the classical argument used to ensure the rationality of the zeta function for a $\mathcal{C}^1$ diffeomorphism defined on a hyperbolic set with local product structure, which profits by the existence of Markov partitions with arbitrarily small diameter. Within the Ruelle-expanding setting, we will prove the existence of a finite cover with analogous properties, which will play the same role the Markov partition did. Moreover, we will see that there is also a relation between the topological entropy and the radius of convergence of the zeta function in this case.

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2 The zeta function

Definition 2.1 If f is a continuous map of a topological space X, let $N_n(f)$ denote the number of periodic points with period n, that is, the points x for which $f^n(x) = x$. If $N_n(f) < \infty, \forall n \in \mathbb{N}$, we define the **zeta function** of f as

$$\zeta_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n(f)}{n} t^n\right)$$

As the exponential is an entire function, the radius of convergence of $\zeta_f(t)$ is given by

$$\rho = \frac{1}{\limsup \sqrt[n]{\frac{N_n(f)}{n}}} = \frac{1}{\limsup \sqrt[n]{N_n(f)}}$$

(since $\lim \sqrt[n]{n} = 1$).

Let $L = -\log \rho$, so that $\rho = e^{-L}$. Then, we have

$$L = -\log \frac{1}{\limsup \sqrt[n]{N_n(f)}} = \limsup(1/n) \log N_n(f)$$

2.1 Examples

2.1.1 Markov subshifts of finite type

Let k be a natural number and $[k]$ the set $\{1, 2, \ldots, k\}$ with the discrete topology. Consider $\Sigma(k)$ the product space $[k]^{\mathbb{Z}}$, whose elements are the sequences $\underline{a} = (\ldots, a_{-1}, a_0, a_1, \ldots)$, with $a_n \in [k], \forall n \in \mathbb{Z}$. This space has a product topology, which can be generated by the metric given by

$$d(\underline{a}, \underline{b}) = \sum_{n=-\infty}^{\infty} \frac{\delta_n(\underline{a}, \underline{b})}{2^{2|n|}}$$

where $\delta_n(\underline{a}, \underline{b})$ is 0 when $a_n = b_n$ and 1 otherwise. Notice that

$$0 \leq d(\underline{a}, \underline{b}) \leq \sum_{n \in \mathbb{Z}} \frac{1}{2^{2|n|}} = 1 + 2 \sum_{n \in \mathbb{N}} \frac{1}{2^{2n}} = \frac{5}{3}$$

and that $d(\underline{a}, \underline{b}) \geq 1 \Leftrightarrow a_0 \neq b_0$, since

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{2^{2|n|}} = 2 \sum_{n \in \mathbb{N}} \frac{1}{2^{2n}} = \frac{2}{3} < 1$$

On $\Sigma(k)$ we have defined a homeomorphism, called *shift*, by

$$(\sigma(\underline{a}))_i = a_{i+1}, i \in \mathbb{Z}$$

This way, σ has a special class of closed invariant sets. Let M_k be the set of $k \times k$ matrices with entries 0 or 1. For each $A \in M_k$, we define $\Sigma_A = \{\underline{a} \in \Sigma(k) : A_{a_i a_{i+1}} = 1\}$, which is a closed invariant subspace of $\Sigma(k)$. The pair (Σ_A, σ_A), where $\sigma_A = \sigma|_{\Sigma_A}$, is called a *subshift of finite type.*

A matrix $A \in M_k$ is said to be *irreducible* if $\forall i, j \in [k], \exists n \in \mathbb{N} : (A^n)_{ij} > 0$. In this case, by the *Perron-Frobenius Theorem*, we know that it has a non-negative simple eigenvalue λ which is greater than

the absolute value of all the others eigenvalues, that is, such that $max_{i\in[k]}|\lambda_i| = \lambda$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all the eigenvalues of A. Besides, its entropy is $\log\lambda$. In particular, the entropy of the full shift $\sigma : \Sigma(k) \to \Sigma(k)$ is $\log k$. (See [2]). For such a σ_A, we can actually compute the zeta function: it is a rational function and L is precisely the entropy of f. Let us recall why.

We say that a finite sequence $a_0a_1...a_n$ of elements in $[k]$ is *admissible* if $A_{a_ia_{i+1}} = 1$. Let $N_n(p, q, A)$ denote the number of admissible sequences of length $n + 1$ which start at p and end at q.

Proposition 3 $N_n(p, q, A) = (A^n)_{pq}$

Proof: We use induction over n. For $n = 1$, this is true by definition of A. Suppose this is true for $n = m - 1$. Then, for $n = m$ we have

$$N_m(p, q, A) = \sum_{r=1}^{k} N_{m-1}(p, r, A)A_{rq} = \sum_{r=1}^{k}(A^{m-1})_{pr}A_{rq} = (A^m)_{pq}$$

and the number of admissible sequences of length $n + 1$ which start and end with the same element of $[k]$ is

$$\sum_{p=1}^{k} N_n(p, p, A) = \sum_{p=1}^{k}(A^n)_{pp} = \mathrm{tr}(A^n)$$

Notice that $\underline{a} \in \Sigma_A$ is a fixed point of σ_A^n if and only if $a_i = a_{i+n}, \forall i \in \mathbb{Z}$. Then, for each fixed point of σ_A^n given by

$$\underline{a} = (..., a_0, a_1, a_2, ..., a_n, a_{n+1}, a_{n+2}, ...) = (..., a_0, a_1, a_2, ..., a_0, a_1, a_2, ...)$$

we can associate a unique admissible sequence of length $n + 1$ given by $a_0a_1a_2...a_{n-1}a_0$. Therefore, the number of fixed points of σ_A^n is $N_n(\sigma_A) = \mathrm{tr}(A^n)$. $\square$

Theorem 3.1 $\zeta_{\sigma_A}(t) = 1/\det(I - tA)$

Proof: Let $\lambda_1, \lambda_2, ..., \lambda_k$ be the eigenvalues of A, so that

$$\det(tI - A) = (t - \lambda_1)(t - \lambda_2)\ldots(t - \lambda_k)$$

Replacing t by t^{-1}, we get

$$\det(t^{-1}I - A) = (t^{-1} - \lambda_1)(t^{-1} - \lambda_2)\ldots(t^{-1} - \lambda_k)$$

and, multiplying both sides by t^k, we get

$$t^k\det(t^{-1}I - A) = t^k(t^{-1} - \lambda_1)(t^{-1} - \lambda_2)\ldots(t^{-1} - \lambda_k)$$
$$\det(I - tA) = (1 - \lambda_1t)(1 - \lambda_2t)\ldots(1 - \lambda_kt)$$

Besides, we have

$$\zeta_{\sigma_A}(t) = \exp\left(\sum_{n=1}^{\infty}\frac{N_n(\sigma_A)}{n}t^n\right) = \exp\left(\sum_{n=1}^{\infty}\frac{\mathrm{tr}(A^n)}{n}t^n\right)$$

Since the eigenvalues of A^n are $\lambda_1^n, \lambda_2^n, ..., \lambda_k^n$, we get $\operatorname{tr}(A^n) = \sum_{m=1}^{k} \lambda_m^n$. So,

$$\zeta_{\sigma_A}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{\sum_{m=1}^{k} \lambda_m^n}{n} t^n\right) = \exp\left(\sum_{m=1}^{k}\left(\sum_{n=1}^{\infty} \frac{(\lambda_m t)^n}{n}\right)\right)$$

Moreover, since $\sum_{n=1}^{\infty} \frac{t^n}{n} = \log\left(\frac{1}{1-t}\right)$, we have

$$\zeta_{\sigma_A}(t) = \exp\left(\sum_{m=1}^{k} \log\left(\frac{1}{1-\lambda_m t}\right)\right) =$$

$$= \exp\left(\log\left(\prod_{m=1}^{k}\left(\frac{1}{1-\lambda_m t}\right)\right)\right) = \frac{1}{\prod_{m=1}^{k}(1-\lambda_m t)} = \frac{1}{\det(I - tA)}$$

$\square$

For instance, if $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, its eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$, so

$$\zeta_{\sigma_A}(t) = \frac{1}{(1-\lambda_1 t)(1-\lambda_2 t)} = \frac{1}{1-t-t^2}$$

Proposition 4 *Let A be an irreducible matrix with entries 0 or 1. Then the topological entropy of σ_A is $-\log \rho$, where ρ is the radius of convergence of ζ_{σ_A}.*

Proof: In fact, since $\zeta_{\sigma_A}(t) = 1/\det(I - tA)$ and

$$\det(I - tA) = 0 \Leftrightarrow \prod_{m=1}^{k}(1-\lambda_m t) = 0 \Leftrightarrow \exists m \in [k] : t = 1/\lambda_m \wedge \lambda_m \neq 0$$

the radius of convergence of ζ_{λ_A} is

$$\rho = \min\{|1/\lambda_i| : i \in [k] \wedge \lambda_i \neq 0\} = 1/\max\{|\lambda_i| : i \in [k] \wedge \lambda_i \neq 0\} = 1/\lambda$$

Then $L = -\log \rho = \log \lambda$, and this value is precisely the topological entropy of σ_A. $\square$

Remark: However, there are closed invariant subsets of $\Sigma(k)$ for which the zeta function for the restriction of σ to those sets is not rational. In fact:

- The set of rational functions defined in a neighborhood of zero of the form $\exp\left(\sum_{n=1}^{\infty} \frac{N_n}{n} t^n\right)$, with $N_n \in \mathbb{Z}, \forall n \in \mathbb{N}$, is countable. In particular, the set of rational functions which are zeta functions for some restriction of σ is countable.
- There is a noncountable collection of closed invariant subsets of $\Sigma(k)$ such that the zeta function for the restriction of σ to those sets is distinct from each other.

Therefore, there is a noncountable collection of closed invariant subsets of $\Sigma(k)$ such that the zeta function for the restriction of σ to those sets is not rational. For example, let $k = 2$ and $S \subseteq \Sigma(2)$ be the set whose elements are the sequences with only one '1' and the periodic sequences with at most one '1' in a minimal period. Then, S is a closed invariant subset of $\Sigma(2)$. Also, the number of periodic points of period n in S is equal to the sum of the divisors of n, $\sigma(n)$, plus one, that is, $N_n(\sigma|_S) = \sigma(n) + 1$ and hence,

$$\zeta_{\sigma|_S}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{\sigma(n)+1}{n} t^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} t^n + \sum_{n=1}^{\infty} \frac{t^n}{n}\right) =$$
$$= \exp\left(-\log(s(t)) - \log(1-t)\right) = \frac{1}{(1-t)s(t)}$$

where $s(t) = 1 - t - t^2 + t^5 + t^7 - t^{12} + t^{15} - \ldots$ is a power series with arbitrarily long sequences of coefficients equal to zero. Since $s(t)$ isn't rational, $\zeta_{\sigma|_S}$ is not rational as well.

2.1.2 Expansive maps

Definition 4.1 Let (X, d) be a metric space and $f : X \to X$ a continuous map. We say that ε is an *expansive constant* for f if

$$d(f^n(x), f^n(y)) \leq \varepsilon, \forall n \in \mathbb{N}_0 \Longrightarrow x = y$$

The map f is called *expansive* if it has an *expansive constant.* If $f : X \to X$ is a homeomorphism, we say that ε is an expansive constant for f (and f is expansive) if

$$d(f^n(x), f^n(y)) \leq \varepsilon, \forall n \in \mathbb{Z} \Longrightarrow x = y$$

This property ensures that the periodic points of f of period n are isolated and the sets $N_n(f)$ are finite, $\forall n \in \mathbb{N}$ (see [2]). Moreover

Proposition 5 *If (X, d) is a compact metric space and $f : X \to X$ is expansive, then ζ_f has a positive radius of convergence.*

Proof: Suppose that f is a continuous map with expansive constant ε. Let $U_1, \ldots, U_r$ be a cover of X with $diam(U_i) \leq \varepsilon, \forall i \in [r]$. For each $x \in X$, let $\phi(x) = (a_0, a_1, a_2, \ldots)$, with $a_n = \min\{i \in [r] : f^n(x) \in U_i\}$. We can see that $\phi(x) = \phi(y) \Rightarrow d(f^n(x), f^n(y)) \leq \varepsilon, \forall n \in \mathbb{N}_0 \Rightarrow x = y$, so ϕ is injective. Also, if x is periodic with period n, then so is $\phi(x)$. Since the number of periodic points in $[r]^{\mathbb{N}_0}$ with period n is r^n, we have $N_n(f) \leq r^n$ and

$$L = \limsup(1/n)\log N_n(f) \leq \log r \Longrightarrow \rho \geq 1/r > 0$$

If f is a homeomorphism with expansive constant ε, then the proof is similar (we associate to each point of X an unique sequence in $[r]^{\mathbb{Z}}$, which is periodic if the point is periodic). $\square$

Since, for each expansive map, there is some $r \in \mathbb{N}$ such that $N_n(f) \leq r^n, \forall n \in \mathbb{N}$, we may also deduce that

Corollary 5.1

$$1 - r\,|t| \leq |\zeta_f(t)| \leq \frac{1}{1 - r\,|t|}$$

Proof:

$$|\zeta_f(t)| = \left|\exp\left(\sum_{n=1}^{\infty} \frac{N_n(f)}{n} t^n\right)\right| = \exp\left(\sum_{n=1}^{\infty} \frac{N_n(f)}{n} Re(t^n)\right) \leq$$

$$\leq \exp\left(\sum_{n=1}^{\infty} \frac{r^n}{n}\,|t^n|\right) = \exp\left(\sum_{n=1}^{\infty} \frac{(r\,|t|)^n}{n}\right) = \exp\left(\log\left(\frac{1}{1 - r\,|t|}\right)\right) = \frac{1}{1 - r\,|t|}$$

and, similarly,

$$|\zeta_f(t)| = \exp\left(\sum_{n=1}^{\infty} \frac{N_n(f)}{n} Re(t^n)\right) \geq \exp\left(\sum_{n=1}^{\infty} \frac{r^n}{n}(-\,|t^n|)\right) = 1 - r\,|t|$$

for all t with $|t| < 1/r$ (recall that $\rho \geq 1/r$).

□

2.1.3 Axiom A diffeomorphisms

Definition 5.1 Let f be a $\mathcal{C}^1$ diffeomorphism defined on a manifold M. A subset $\Lambda \subseteq M$ is *hyperbolic* if it is compact, f-invariant ($f(\Lambda) = \Lambda$) and there is a decomposition $T_\Lambda M = E^s_\Lambda \oplus E^u_\Lambda$ such that

$$D_x f(E^s_x) = E^s_{f(x)}, \forall x \in \Lambda$$
$$D_x f(E^u_x) = E^u_{f(x)}, \forall x \in \Lambda$$
$$\exists c > 0, \lambda \in]0,1[: \forall x \in \Lambda, \forall n \geq 0,$$

$$\|D_x f^n(v)\| \leq c\lambda^n \|v\|, \forall v \in E^s_x \text{ and } \|D_x f^{-n}(v)\| \leq c\lambda^n \|v\|, \forall v \in E^u_x$$

For each $x \in \Lambda$, these expanding and contracting subbundles are tangent to the stable and unstable submanifolds,

$$W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0\}$$
$$W^u(x) = \left\{y \in M : d(f^{-n}(x), f^{-n}(y)) \to 0\right\}$$

Besides, for small ε, the local submanifolds

$$W^s_\varepsilon(x) = \{y \in M : d(f^n(x), f^n(y)) < \varepsilon, \forall n \geq 0\}$$
$$W^u_\varepsilon(x) = \left\{y \in M : d(f^{-n}(x), f^{-n}(y)) < \varepsilon, \forall n \geq 0\right\}$$

are $\mathcal{C}^1$ disks embedded in M and there is $\delta > 0$ such that, if the distance between two points x and y in Λ is less then δ, then $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ intersect transversely at an unique point, denoted by $[x, y]$.

In particular, if $y = x$ then $W^s_\varepsilon(x) \cap W^u_\varepsilon(x) = \{x\}$, which means that ε is an expansive constant for f (see [1]). We say that Λ has a *local product structure* if $[x, y] \in \Lambda, \forall x, y \in \Lambda$.

If f is a $\mathcal{C}^1$ diffeomorphism defined on a hyperbolic set with local product structure, then f is expansive, so $N_n(f) < \infty, \forall n \in \mathbb{N}$ and we can define the zeta function for f. And moreover, as proved in [1],

Theorem 5.1 *The zeta function of a $\mathcal{C}^1$ diffeomorphism on a hyperbolic set with local product structure is rational.*

As a consequence, if f is a $\mathcal{C}^1$ diffeomorphism such that $\overline{Per(f)}$ is hyperbolic, then $\zeta_f(t)$ is a rational function: in fact, it is known that, if $\overline{Per(f)}$ is hyperbolic, then it has a local product structure; and $\zeta_f(t) = \zeta_{f|_{\overline{Per(f)}}}(t)$. In particular, if f is Axiom A ($\Omega(f)$ is hyperbolic and $\Omega(f) = \overline{Per(f)}$, where $\Omega(f)$ denotes the set of non-wandering points of f), then $\zeta_f(t)$ is rational.

The main ingredient of the classical argument to prove this theorem is the existence of a Markov partition of arbitrarily small diameter, which establishes a codification of most of the orbits of f through a subshift of finite type (for which we already know how to count the periodic points), and a sharp way to translate the properties of the zeta function from the subshift to the diffeomorphism setting.

6 Ruelle-expanding maps

Here, we will explain the nature of another class of maps, called *Ruelle-expanding*, whose zeta function is rational.

Definition 6.1 Let (K, d) be a compact metric space and $f : K \to K$ a continuous map. We say that f is *Ruelle-expanding* if there are $r > 0$, $0 < \lambda < 1$ and $c > 0$ such that:

- $\forall x, y \in K, x \neq y \wedge f(x) = f(y) \Longrightarrow d(x, y) > c$
- $\forall x \in K, \forall a \in f^{-1}(\{x\}), \exists \phi : B_r(x) \to K$ with
  $\phi(x) = a$
  $(f \circ \phi)(y) = y, \forall y \in B_r(x)$
  $d(\phi(y), \phi(z)) \leq \lambda d(y, z), \forall y, z \in B_r(x)$

Examples

- Let M be a compact manifold and $f : M \to M$ a $\mathcal{C}^1$ map. We say that f is *expanding* if $\exists \lambda \in]0, 1[: \forall x \in M, \|D_x f(v)\| \geq 1/\lambda \, \|v\|$. It can be proved (see [3]) that, in this particular case, this condition is equivalent to the previous two from the last definition. So, f is expanding if and only if it is Ruelle-expanding.

  One example of such a map is the application
  $\begin{array}{rcl} f : S^1 & \to & S^1 \\ z & \mapsto & z^k \end{array}$, with $k \in \mathbb{Z}$ and $k > 1$

  (it is easy to see that f is expanding, with $\lambda = 1/k$). Notice that, for this map, we have $N_n(f) = k^n - 1$. So,

$$\zeta_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{k^n - 1}{n} t^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{(kt)^n}{n} - \sum_{n=1}^{\infty} \frac{t^n}{n}\right) =$$

$$= \exp\left(\log\left(\frac{1}{1-kt}\right) - \log\left(\frac{1}{1-t}\right)\right) = \exp\left(\log\left(\frac{1-t}{1-kt}\right)\right) = \frac{1-t}{1-kt}$$

which is a rational function (with a pole at $\frac{1}{k}$, so $\rho = \frac{1}{k} = \exp(-\log(k)) = \exp(-h(f))$).

- Let $\Sigma(k)^+$ be the product space $[k]^{\mathbb{N}_0}$, whose elements are the sequences $\underline{a} = (a_0, a_1, \ldots)$, with $a_n \in [k], \forall n \in \mathbb{N}_0$. As its bilateral version, this space has a product topology which can be generated by the metric given by

$$d(\underline{a}, \underline{b}) = \sum_{n=0}^{\infty} \frac{\delta_n(\underline{a}, \underline{b})}{2^n}$$

where $\delta_n(\underline{a}, \underline{b})$ is 0 when $a_n = b_n$ and 1 otherwise. The *unilateral shift* is the map of $\Sigma(k)^+$ given by

$$(\sigma^+(\underline{a}))_i = a_{i+1}, i \in \mathbb{N}_0$$

For each $A \in M_k$, we define $\Sigma_A^+ = \{\underline{a} \in \Sigma(k)^+ : A_{a_i a_{i+1}} = 1\}$. The pair (Σ_A^+, σ_A^+), where $\sigma_A^+ = \sigma^+|_{\Sigma_A^+}$, is called a *unilateral subshift of finite type*. If A is irreducible, then it is easy to see that σ_A^+ is Ruelle-expanding, with $r = 1$ and $\lambda = c = 1/2$, since:

- If $\underline{a} \neq \underline{b}$ and $\sigma_A^+(\underline{a}) = \sigma_A^+(\underline{b})$, then $a_0 \neq b_0$, so $d(\underline{a}, \underline{b}) \geq 1 > c$.

- If $r = 1$, then, for any $\underline{a} \in \Sigma_A^+$ we have $B_r(\underline{a}) = \{\underline{b} \in \Sigma_A^+ : b_0 = a_0\}$ since, as we have seen, $b_0 \neq a_0 \Rightarrow d(\underline{a}, \underline{b}) \geq 1 = r$. Also, the pre-images of $\underline{a} = (a_0, a_1, a_2, \ldots)$ are of the form $(x, a_0, a_1, \ldots)$, where $A_{x a_0} = 1$ (there is at least one $x \in [k]$ such that $A_{x a_0} = 1$ because A is irreducible). If we define $\phi(\underline{b}) = (x, b_0, b_1, b_2, \ldots)$ for $\underline{b} = (b_0, b_1, b_2, \ldots) \in B_r(\underline{a})$ (that is, with $a_0 = b_0$), then $\sigma_A^+(\phi(\underline{b})) = \underline{b}$ and

$$d(\phi(\underline{b}), \phi(\underline{c})) = \sum_{n=1}^{\infty} \frac{\delta_{n-1}(\underline{b}, \underline{c})}{2^n} = \sum_{n=0}^{\infty} \frac{\delta_n(\underline{b}, \underline{c})}{2^{n+1}} = \frac{d(\underline{b}, \underline{c})}{2} = \lambda d(\underline{b}, \underline{c}), \forall \underline{b}, \underline{c} \in B_r(\underline{a})$$

If, to simplify the notation, we denote by σ the map σ_A^+, then $\underline{a} \in \Sigma_A^+$ is a fixed point of σ^n if and only if $a_i = a_{i+n}, \forall i \in \mathbb{N}_0$. For each fixed point of σ^n given by

$$\underline{a} = (a_0, a_1, a_2, ..., a_n, a_{n+1}, a_{n+2}, ...) = (a_0, a_1, a_2, ..., a_0, a_1, a_2, ...)$$

we can associate a unique admissible sequence of length $n+1$ given by $a_0 a_1 a_2 ... a_{n-1} a_0$. So, the number of fixed points of σ^n is $N_n(\sigma) = \operatorname{tr}(A^n)$ and $\zeta_\sigma(t) = 1/\det(I - tA)$, also a rational function (with poles at the inverses of the eigenvalues of A).

Definition 6.2 Let $f : K \to K$ be Ruelle-expanding and $S \subseteq K$. Given $n \in \mathbb{N}$, we say that $g : S \to K$ is a *contractive branch* of f^{-n} if

- $(f^n \circ g)(x) = x, \forall x \in S$
- $d((f^j \circ g)(x), (f^j \circ g)(y)) \leq \lambda^{n-j} d(x, y), \forall x, y \in S, j \in \{0, 1, \ldots, n\}$

It is easy to see (details in [3]) that, given $x \in K$ and $a \in f^{-n}(\{x\})$ for some $n \in \mathbb{N}$, there is always a contractive branch $g : B_r(x) \to K$ of f^{-n} with $g(x) = a$. Moreover,

Proposition 7 *Let $B(n, \varepsilon, x) = \{y \in K : d(f^j(x), f^j(y)) < \varepsilon, \forall j \in \{0, \ldots, n\}\}$. There is some $\varepsilon_0 < r$ such that, for every ε with $0 < \varepsilon < \varepsilon_0$, we have*

- *$\forall n \in \mathbb{N}$, $B(n, \varepsilon, x) = g(B_\varepsilon(f^n(x)))$, where $g : B_r(f^n(x)) \to K$ is a contractive branch of f^{-n} with $g(f^n(x)) = x$*
- *ε is an expansive constant for f*

Proof: See [3]. □

Proposition 8 *$K = \bigcup_{n \geq 0} f^{-n}(\overline{Per(f)})$, where $Per(f)$ is the set of periodic points for f. In particular, $Per(f) \neq \emptyset$.*

Proof: See [3]. □

Notice that, since f is expansive, we can consider the zeta function for f and, as $Per(f) \neq \emptyset$, given $x \in K$ with $f^k(x) = x$ for some $k \in \mathbb{N}$, we have $f^{nk}(x) = x$ and $N_{nk}(f) \geq 1, \forall n \in \mathbb{N}$, which implies that $L \geq \limsup \frac{1}{nk} \log N_{nk}(f) \geq 0$ and $\rho \leq 1$.

Is there any relation between L and $h(f)$ if f is Ruelle-expanding? In fact, we have that $L \leq h(f)$ but, to prove it, we need to simplify the calculus of $h(f)$. Let us first recall briefly how to evaluate, in general, this number.

Given a metric space (X, d) and a uniformly continuous map $f : X \to X$, for every $n \in \mathbb{N}$, we define a dynamical metric d_n on X by

$$d_n(x, y) = \max\{d(f^i(x), f^i(y)), i \in \{0, 1, \ldots, n-1\}\}$$

and the corresponding open dynamical ball, with center x and radius r,

$$B(n-1, r, x) = \{y \in K : d(f^j(x), f^j(y)) < r, \forall j \in \{0, \ldots, n-1\}\} = \bigcap_{i=0}^{n-1} f^{-i}(B_r(f^i(x)))$$

and closed dynamical ball

$$\overline{B}(n-1, r, x) = \{y \in K : d(f^j(x), f^j(y)) \leq r, \forall j \in \{0, \ldots, n-1\}\} = \bigcap_{i=0}^{n-1} f^{-i}(\overline{B}_r(f^i(x)))$$

Accordingly,

Definition 8.1 Let $n \in \mathbb{N}$, $\varepsilon > 0$ and K be a compact subset of X. Given a subset F of X, we say that F $(n, \varepsilon)-$ *spans* K with respect to f if

$$\forall x \in K, \exists y \in F : d_n(x, y) \leq \varepsilon$$

or, equivalently,

$$K \subseteq \bigcup_{y \in F} \overline{B}(n-1, \varepsilon, y)$$

Definition 8.2 Let $n \in \mathbb{N}$, $\varepsilon > 0$ and K be a compact subset of X. We define $r_n(\varepsilon, K)$ as the smallest cardinality of any (n, ε) spanning set for K with respect to f.

Notice that, since K is compact, we have $r_n(\varepsilon, K) < \infty$; and $\varepsilon_1 < \varepsilon_2 \Longrightarrow r_n(\varepsilon_1, K) \geq r_n(\varepsilon_2, K)$.

Definition 8.3 Let $\varepsilon > 0$ and K be a compact subset of X. Then

$$r(\varepsilon, K) = r(\varepsilon, K, f) = \limsup_{n \to \infty} (1/n) \log r_n(\varepsilon, K)$$

Definition 8.4 If for each compact subset K of X we denote by $h(f, K)$ the limit $\lim_{\varepsilon \to 0} r(\varepsilon, K, f)$, then the *topological entropy* of f is $h(f) = \sup\{h(f, K), K \text{ compact subset of } X\}$.

Sometimes it is useful to use an equivalent way of defining topological entropy which uses *separated sets* instead of spanning ones.

Definition 8.5 Let $n \in \mathbb{N}$, $\varepsilon > 0$ and K be a compact subset of X. Given a subset E of K, we say that *E is (n, ε) separated with respect to f* if

$$\forall x, y \in E, d_n(x, y) \leq \varepsilon \Longrightarrow x = y$$

or, equivalently,

$$\forall x \in E, \overline{B}(n-1, \varepsilon, x) = \{x\}$$

Definition 8.6 Let $n \in \mathbb{N}$, $\varepsilon > 0$ and K be a compact subset of X. We define $s_n(\varepsilon, K)$ as the largest cardinality of any (n, ε) separated set for K with respect to f.

Observe that $r_n(\varepsilon, K) \leq s_n(\varepsilon, K) \leq r_n(\varepsilon/2, K)$ and so, since $r_n(\varepsilon/2, K) < \infty$, we have $s_n(\varepsilon, K) < \infty$; besides, $\varepsilon_1 < \varepsilon_2 \Longrightarrow s_n(\varepsilon_1, K) \geq s_n(\varepsilon_2, K)$.

Definition 8.7 Let $\varepsilon > 0$ and K be a compact subset of X. We define

$$s(\varepsilon, K) = s(\varepsilon, K, f) = \limsup_{n \to \infty} (1/n) \log s_n(\varepsilon, K)$$

As a consequence of the previous inequalities, we get $r(\varepsilon, K) \leq s(\varepsilon, K) \leq r(\varepsilon/2, K)$ and so (see [2])

Proposition 9 *(a) For any compact subset K of X, we have $h(f, K) = \lim_{\varepsilon \to 0} s(\varepsilon, K)$.*

(b) $h(f) = \sup_K h(f, K) = \sup_K \lim_{\varepsilon \to 0} s(\varepsilon, K, f)$.

(c) In case X is compact, then
$h(f) = h(f, X) = \lim_{\varepsilon \to 0} \limsup (1/n) \log r_n(\varepsilon, X) = \lim_{\varepsilon \to 0} \limsup (1/n) \log s_n(\varepsilon, X)$.

Let us now go back to Ruelle-expanding maps.

Proposition 10 *If $f : X \to X$ is a Ruelle-expanding map of a compact metric space (X, d), then $h(f) = r(\varepsilon_0, X) = s(\varepsilon_0, X)$ for all $\varepsilon_0 < \varepsilon/4$, where ε is an expansive constant for f.*

Proof: See [2]. (Although the proof is for expansive homeomorphisms, it can be easily adapted for expansive maps.) $\square$

Corollary 10.1 *For any Ruelle-expanding map we have $L \leq h(f)$, that is, the radius of convergence of the zeta function is $\rho \geq \exp(-h(f))$.*

Proof: Let p and q be periodic points of f, with $f^n(p) = p$ and $f^n(q) = q$ for some $n \in \mathbb{N}$. Then, we have

$$d_n(p,q) \leq \varepsilon_0 \Longrightarrow d_n(p,q) \leq \varepsilon \Longrightarrow d(f^i(p), f^i(q)) \leq \varepsilon, \forall i \in \{0,1,\ldots,n-1\} \Longrightarrow$$
$$\Longrightarrow d(f^i(p), f^i(q)) \leq \varepsilon, \forall i \in \mathbb{N}_0 \Longrightarrow p = q$$

So, the set P_n of periodic points p with $f^n(p) = p$ is a (n, ε_0) separated set for X and $s_n(\varepsilon_0, X) \geq card(P_n) = N_n(f)$. Consequently,

$$L = \limsup(1/n) \log N_n(f) \leq \limsup(1/n) \log s_n(\varepsilon_0, X) = s(\varepsilon_0, X) = h(f)$$

$\square$

This yields a link between $h(f)$ and the number of pre-images of the points in X by f.

Lemma 10.1 *If (X,d) is a compact metric space and $f : X \to X$ is a Ruelle-expanding map, then there is a $k \in \mathbb{N}$ such that $card(f^{-1}(\{x\})) \leq k, \forall x \in X$.*

Proof: If we set $E = f^{-1}(\{x\})$ then we have $f(u) = f(v) = x, \forall u, v \in E, u \neq v$, so $d_1(u,v) = d(u,v) > c$ and E is a $(1,c)$ separated set. Since $card(E) \leq s_1(c,X) < \infty$, we can take $k = s_1(c,X)$. $\square$

Proposition 11 *$h(f) \leq \log(k)$, with equality if $card(f^{-1}(\{x\})) = k, \forall x \in X$.*

Proof: Let $\varepsilon_0 < min\{\varepsilon/4, c, r\}$. Since X is compact, there is a finite set F for which we can write

$$X = \bigcup_{y \in F} \overline{B}_{\varepsilon_0}(y)$$

Given $x \in X$ and $n \in \mathbb{N}$, let $y \in F$ be such that $d(f^n(x), y) \leq \varepsilon_0$ and let $g : B_r(f^n(x)) \to X$ be a contractive branch of f^{-n} with $g(f^n(x)) = x$. If we take $z = g(y)$, we have

- $f^n(z) = f^n(g(y)) = y \Longrightarrow z \in f^{-n}(F)$
- $d(f^i(x), f^i(z)) = d(f^i(g(f^n(x))), f^i(g(y))) \leq \lambda^{n-i} d(f^n(x), y) \leq \lambda^{n-i}\varepsilon_0 \leq \varepsilon_0, \forall i \in \{0,1,\ldots,n-1\} \Longrightarrow d_n(x,z) \leq \varepsilon_0$

So, $f^{-n}(F)$ is a (n, ε_0) spanning set for X. Therefore, $r_n(\varepsilon_0, X) \leq card(f^{-n}(F)) \leq k^n card(F), \forall n \in \mathbb{N}$ and we get

$$h(f) = r(\varepsilon_0, X) = \limsup(1/n) \log r_n(\varepsilon_0, X) \leq \limsup(1/n) \log(k^n card(F)) =$$
$$= \limsup(\log k + (1/n) \log(card(F))) = \log k$$

As a consequence, we have $0 \leq L \leq \log k$ and $1/k \leq \rho \leq 1$.

Suppose now that there is some $k \in \mathbb{N}$ such that $card(f^{-1}(\{x\})) = k, \forall x \in X$. Take a point $x \in X$. If we consider $E_n = f^{-n}(\{x\})$, then we have $f^n(u) = f^n(v) = x, \forall u, v \in E_n, u \neq v$. If $f(u) = f(v)$, then $d_n(u,v) \geq d(u,v) > c$, otherwise, we have $f(u) \neq f(v)$. Admitting the last case, if $f^2(u) = f^2(v)$, then $d_n(u,v) \geq d(f(u), f(v)) > c$, otherwise, we have $f^2(u) \neq f^2(v)$. Proceeding, and since we have $f^n(u) = f^n(v)$, there must be some $j \in \{1, \ldots, n\}$ for which $f^j(u) = f^j(v)$ and $f^{j-1}(u) \neq f^{j-1}(v)$, so $d_n(u,v) \geq d(f^{j-1}(u), f^{j-1}(v)) > c$ and E_n is a (n, c) separated set. Since $card(E_n) = k^n$, we have $k^n \leq s_n(c, X) \leq s_n(\varepsilon_0, X)$ and we get

$$h(f) = s(\varepsilon_0, X) = \limsup(1/n) \log s_n(\varepsilon_0, X) \geq \limsup(1/n) \log(k^n) = \log k$$

which allow us to conclude that, in this particular case, $h(f) = \log k$.

□

Now, our goal will be to prove the rationality of the zeta function for Ruelle-expanding maps. Recall that the existence of a Markov partition was an essential ingredient in the proof of the rationality of the zeta function for $\mathcal{C}^1$ diffeomorphisms defined on a hyperbolic set with local product structure. In the case of Ruelle-expanding maps, we will prove the existence of a finite cover with analogous properties, which will play the same role the Markov partition did.

Proposition 12 *Let f be a Ruelle-expanding map defined on a compact set K. Let ε denote an expansive constant for f. Then, K has a finite cover $\{R_1, ..., R_n\}$ with the following properties:*

- *Each R_i has a diameter less than $min\{\varepsilon, c/2\}$ and is proper, that is, equal to the closure of its interior.*
- $\overset{\circ}{R_i} \cap \overset{\circ}{R_j} = \emptyset, \forall i, j \in [n], i \neq j$
- $f(\overset{\circ}{R_i}) \cap \overset{\circ}{R_j} \neq \emptyset \Longrightarrow \overset{\circ}{R_j} \subseteq f(\overset{\circ}{R_i})$

Remark: If $\overset{\circ}{R_j} \subseteq f(\overset{\circ}{R_i})$, then $R_j = \overline{\overset{\circ}{R_j}} \subseteq \overline{f(\overset{\circ}{R_i})} \subseteq f\left(\overline{\overset{\circ}{R_i}}\right) = f(R_i)$ and the last condition means that

$f(\overset{\circ}{R_i}) \cap \overset{\circ}{R_j} \neq \emptyset \Longrightarrow R_j \subseteq f(R_i)$

To prove this proposition, we will begin by a shadowing lemma.

Lemma 12.1 *Let $f : K \to K$ be Ruelle-expanding. For any $\beta \in]0, r[$ there is some $\alpha > 0$ such that, if $(x_n)_{n \in \mathbb{N}_0}$ is a α-pseudo orbit in K (that is, if $d(f(x_n), x_{n+1}) < \alpha, \forall n \in \mathbb{N}_0$), then it admits a β-shadow (that is, a point $x \in K$ such that $d(f^n(x), x_n) < \beta, \forall n \in \mathbb{N}_0$). Besides, the β-shadow is unique if $\beta < \varepsilon/2$, where ε is an expansive constant for f.*

Proof: We will start proving this assertion for finite α-pseudo orbits. Let $\beta \in]0, r[$ and $(x_0, x_1, \ldots, x_n)$ be such that $d(f(x_{k-1}), x_k) < \alpha, \forall k \in [n]$ for some $\alpha > 0$. If $y_n = x_n$, then $d(y_n, x_n) = 0 < \beta$. Now, suppose that $d(y_k, x_k) < \beta$ for $k \in [n]$. Since $d(f(x_{k-1}), x_k) < \alpha$, we have $d(y_k, f(x_{k-1})) < \alpha + \beta < r$ if we assume $\alpha < r - \beta$. Then, we can take $y_{k-1} = g(y_k)$, where $g : B_r(f(x_{k-1})) \to K$ is a contractive branch of f^{-1} with $g(f(x_{k-1})) = x_{k-1}$, and we have $d(y_{k-1}, x_{k-1}) \leq \lambda d(y_k, f(x_{k-1})) < \lambda(\alpha + \beta) < \beta$ if we assume $\alpha < \frac{1-\lambda}{\lambda}\beta$. Also, notice that $y_k = f(y_{k-1}), \forall k \in [n]$, so that $y_k = f^k(x), \forall k \in [n]$ for $x = y_0$. Hence, it suffices to take $\alpha < \min\{r - \beta, \frac{1-\lambda}{\lambda}\beta\}$.

Now, take $\beta \in]0, r[$ and let $(x_n)_{n\in\mathbb{N}_0}$ be a α-pseudo orbit, with $\alpha < \min\{r - \beta/2, \frac{1-\lambda}{\lambda}\beta/2\}$. Let z_n be a $\beta/2$-shadow of $(x_0, x_1, \ldots, x_n)$; since K is compact, there is some subsequence $(z_{n_k})_k$ converging to some point $z \in K$. We have $d(f^i(z_{n_k}), x_i) < \beta/2, \forall i \in \{0, 1, \ldots, n_k\}$, so, for $i \in \mathbb{N}_0$ fixed we get $d(f^i(z), x_i) = \lim d(f^i(z_{n_k}), x_i) \leq \beta/2 < \beta$ and we conclude that z is a β-shadow of $(x_n)_{n\in\mathbb{N}_0}$.

For the uniqueness of the β-shadow when $\beta < \varepsilon/2$, suppose that z and z' are both β-shadows of $(x_n)_{n\in\mathbb{N}_0}$. Then, we have $d(f^i(z), f^i(z')) \leq d(f^i(z), x_i) + d(f^i(z'), x_i) < 2\beta < \varepsilon, \forall i \in \mathbb{N}_0$, so $z = z'$.

□

Let ε be an expansive constant for f with $\varepsilon < r$ and fix some $\beta < \min\{\varepsilon/2, c/4\}$. Let α be given by the previous lemma and $\gamma \in]0, \alpha/2[$ be such that $d(x, y) < \gamma \Rightarrow d(f(x), f(y)) < \alpha/2, \forall x, y \in K$. Since K is compact, we can take $\{p_1, \ldots, p_k\}$ such that $K = \bigcup_{i=1}^k B_\gamma(p_i)$. We define a matrix $A \in M_k$ by

$$A_{ij} = 1 \text{ if } d(f(p_i), p_j) < \alpha \text{ and } A_{ij} = 0 \text{ otherwise.}$$

For every $\underline{a} \in \Sigma_A^+$ the sequence $(p_{a_i})_{i\in\mathbb{N}_0}$ is a α-pseudo orbit, so it admits an unique β-shadow which we will denote by $\theta(\underline{a})$. Therefore, we have defined a map $\theta : \Sigma_A^+ \to K$.

Lemma 12.2 *θ is a semiconjugacy of σ_A^+ and f, that is, θ is surjective, continuous and verifies $f \circ \theta = \theta \circ \sigma_A^+$.*

Proof: Given $x \in K$, we can take $a_i \in [k]$ so that $d(f^i(x), p_{a_i}) < \gamma$ for any $i \in \mathbb{N}_0$; then, $d(f(p_{a_i}), p_{a_{i+1}}) \leq d(f(p_{a_i}), f(f^i(x))) + d(f^{i+1}(x), p_{a_{i+1}}) < \alpha/2 + \gamma < \alpha$ and $(p_{a_i})_{i\in\mathbb{N}_0}$ is a α-pseudo orbit. So, $x = \theta(\underline{a})$ and θ is surjective.

For the continuity, since K is compact it suffices to see that, for any two sequences $(\underline{s}^n)_{n\in\mathbb{N}}$ and $(\underline{t}^n)_{n\in\mathbb{N}}$ converging to the same limit l in Σ_A^+ whose images under θ converge respectively to s and t in K, we have $s = t$. Fix some $i \in \mathbb{N}_0$; for any $n \in \mathbb{N}$, we have $d(f^i(\theta(\underline{s}^n)), p_{s_i^n}) < \beta$ and $d(f^i(\theta(\underline{t}^n)), p_{t_i^n}) < \beta$. So, taking limits we have $d(f^i(s), p_{l_i}) \leq \beta$ and $d(f^i(t), p_{l_i}) \leq \beta$. Hence, $d(f^i(s), f^i(t)) \leq 2\beta < \varepsilon$ and, since ε is an expansive constant for f, we get $s = t$.

Finally, the relation $f \circ \theta = \theta \circ \sigma_A^+$ is a consequence of the unicity of the β-shadow and the fact that, if x is a β-shadow for $(p_{a_i})_i$, then $f(x)$ is a β-shadow for $(p_{a_{i+1}})_i = (p_{\sigma_A^+(a_i)})_i$.

□

Let $T_i = \{\theta(\underline{a}) : a_0 = i\}$ for $i \in [k]$. Then, $T_i = \theta(C_i)$ where $C_i = \{\underline{a} \in \Sigma_A^+ : a_0 = i\}$ and, since $\Sigma_A^+ = \bigcup_{i=1}^k C_i$, we have $K = \bigcup_{i=1}^k T_i$ because θ is surjective. Hence, $\{T_i, i \in [k]\}$ is a finite closed cover of K (T_i is closed since C_i is compact and θ is continuous).

Lemma 12.3 *If $A_{ij} = 1$, then $T_j \subseteq f(T_i)$ and $\overset{\circ}{T_j} \subseteq f(\overset{\circ}{T_i})$. Also, given $x \in T_i$ with $f(x) \in T_j$, if $g : B_r(f(x)) \to K$ is a contractive branch of f^{-1} with $g(f(x)) = x$, then $g(T_j) \subseteq T_i$ and $g(\overset{\circ}{T_j}) \subseteq \overset{\circ}{T_i}$.*

Proof: Given any $y \in T_j$, we have $y = \theta(\underline{b})$ for some $\underline{b} \in \Sigma_A^+$ with $b_0 = j$. Since $A_{ij} = 1$, we can take $\underline{c} = (i, b_0, b_1, b_2, \ldots) \in \Sigma_A^+$, and so $y = \theta(\underline{b}) = \theta(\sigma_A^+(\underline{c})) = f(\theta(\underline{c})) \in f(\theta(C_i)) = f(T_i)$. Then, $T_j \subseteq f(T_i)$

Notice that $T_j \subseteq B_\beta(p_j)$. Since $d(f(x), p_j) < \beta$, we have $T_j \subseteq B_{2\beta}(f(x)) \subseteq B_r(f(x))$. Let $g : B_r(f(x)) \to K$ be a contractive branch of f^{-1} with $g(f(x)) = x$. Given $y \in T_j$, we have $y = f(z)$ for some $z \in T_i$. Then,

$$d(g(y), z) \leq d(g(y), g(f(x))) + d(x, p_i) + d(p_i, z) < d(y, f(x)) + 2\beta < 4\beta < c$$

and, since $f(g(y)) = y = f(z)$, we get $g(y) = z \in T_i$. So, $g(T_j) \subseteq T_i$.

It is easy to see that $g : B_r(f(x)) \to g(B_r(f(x)))$ is a homeomorphism, with $g^{-1} = f|_{g(B_r(f(x)))}$: $g(B_r(f(x))) \to B_r(f(x))$. Therefore, we conclude that $g(\overset{\circ}{T_j}) = \overset{\circ}{\overbrace{g(T_j)}} \subseteq \overset{\circ}{T_i}$ and $\overset{\circ}{T_j} = f(g(\overset{\circ}{T_j})) \subseteq f(\overset{\circ}{T_i})$.

□

Let $Z = K \backslash \bigcup_{i=1}^{k} \partial T_i$. Notice that, since T_i is a closed set, ∂T_i has empty interior. So, Z is dense in K. Given $x \in Z$ we define

$$T_i^*(x) = \overset{\circ}{T_i} \text{ if } x \in \overset{\circ}{T_i}$$
$$T_i^*(x) = K \backslash T_i \text{ if, otherwise, } x \notin T_i$$
$$R(x) = \bigcap_{i=1}^{k} T_i^*(x)$$

The sets $R(x)$ satisfy the following properties:

- $R(x)$ is open (because it is a finite intersection of open sets)
- $x \in R(x)$ (because $x \in T_i^*(x), \forall i \in [k]$)
- $R(x) \subseteq \overset{\circ}{T_i}$ for some $i \in [k]$

  (since $\bigcap_{i=1}^{k} K \backslash T_i = K \backslash \bigcup_{i=1}^{k} T_i = \emptyset$, we must have $x \in \overset{\circ}{T_i}$ for some $i \in [k]$)

- If $R(x) \cap R(y) \neq \emptyset$, then $R(x) = R(y)$

  (in fact, we have $R(x) \cap R(y) \neq \emptyset \Rightarrow \forall i \in [k], T_i^*(x) \cap T_i^*(y) \neq \emptyset \Rightarrow \forall i \in [k], T_i^*(x) = T_i^*(y) \Rightarrow R(x) = R(y)$)

Moreover,

Lemma 12.4 *Given $x \in Z \cap f^{-1}(Z)$, we have $g(R(f(x))) \subseteq R(x)$, where $g : B_r(f(x)) \to K$ is a contractive branch of f^{-1} with $g(f(x)) = x$.*

Proof: Let $y \in R(f(x))$. Notice that $y \in Z$ and $f(x) \in R(y)$.

For $i \in [k]$, if $x \in T_i$ then $x = \theta(\underline{a})$ for some $\underline{a} \in \Sigma_A^+$ with $a_0 = i$. Let $j = a_1$. Then, $f(x) = \theta(\sigma(\underline{a}))$ and $f(x) \in T_j$, so that $y \in R(f(x)) \subseteq T_j \Rightarrow g(y) \in g(T_j)$. Since $A_{ij} = 1$, by the previous lemma we get $g(T_j) \subseteq T_i$ and, hence, $g(y) \in T_i$.

On the other hand, if $g(y) \in T_i$ then $g(y) = \theta(\underline{b})$ for some $\underline{b} \in \Sigma_A^+$ with $b_0 = i$. Let $j = b_1$. Then, $y = f(g(y)) = \theta(\sigma(\underline{b}))$ and $y \in T_j$, so that $f(x) \in R(y) \subseteq T_j \Rightarrow x = g(f(x)) \in g(T_j)$. Since $A_{ij} = 1$, by the previous lemma we get $g(T_j) \subseteq T_i$ and, hence, $x \in T_i$. So, $x \in T_i \Leftrightarrow g(y) \in T_i, \forall i \in [k]$.

Similarly, using the previous lemma we get $x \in \mathring{T}_i \Leftrightarrow g(y) \in \mathring{T}_i, \forall i \in [k]$. This way, we conclude that $g(y) \in R(x)$.

$\square$

Let $R = \{\overline{R(x)}, x \in Z\}$. Since R is obviously a finite set, we can write $R = \{R_1, \ldots, R_s\}$ with $R_i \neq R_j$ if $i \neq j$. Also, since Z is dense in K, we have $K = \overline{\bigcup_{x\in Z}\{x\}} = \overline{\bigcup_{x\in Z} R(x)} = \bigcup_{x\in Z} \overline{R(x)} = \bigcup_{i=1}^{s} R_i$, that is, R is a finite closed cover of K. Let us see that R satisfies the required properties.

1. R_i has a diameter less than $min\{\varepsilon, c/2\}$ and is proper.

   Take $x \in Z$ such that $R_i = \overline{R(x)}$ and $j \in [k]$ such that $R(x) \subseteq \mathring{T}_j$. Then, $R_i = \overline{R(x)} \subseteq \overline{\mathring{T}_j} \subseteq \overline{T_j} = T_j$ and $diam(R_i) \leq diam(T_j) \leq 2\beta < min\{\varepsilon, c/2\}$. Also, using the fact that the closure of the interior of the closure of the interior of a set is just the closure of the interior of that set, we have

   $\overline{\mathring{R}_i} = \overline{\mathring{\overline{R(x)}}} = \overline{\mathring{\overline{\mathring{R}(x)}}} = \overline{\mathring{R}(x)} = \overline{R(x)} = R_i$ because $R(x)$ is open.

2. $\mathring{R}_i \cap \mathring{R}_j = \emptyset, \forall i, j \in [n], i \neq j$

   Take $x, y \in Z$ such that $R_i = \overline{R(x)}$ and $R_j = \overline{R(y)}$. Suppose that $\mathring{R}_i \cap \mathring{R}_j \neq \emptyset$; using the fact that any open set that intersects the closure of a set also intersects the set itself, we get

   $\mathring{\overline{R(x)}} \cap \mathring{\overline{R(y)}} \neq \emptyset \Rightarrow \mathring{\overline{R(x)}} \cap \overline{R(y)} \neq \emptyset \Rightarrow \mathring{\overline{R(x)}} \cap R(y) \neq \emptyset \Rightarrow \overline{R(x)} \cap R(y) \neq \emptyset \Rightarrow$
   $\Rightarrow R(x) \cap R(y) \neq \emptyset \Rightarrow R(x) = R(y) \Rightarrow R_i = R_j \Rightarrow i = j$

3. $f(\mathring{R}_i) \cap \mathring{R}_j \neq \emptyset \Rightarrow \mathring{R}_j \subseteq f(\mathring{R}_i)$

   Since f takes open sets into open sets and Z is dense in K, $f^{-1}(Z)$ is also dense in K. Also, Z is a nonempty open set, so $Z \cap f^{-1}(Z)$ is dense in Z, and, hence, $Z \cap f^{-1}(Z)$ is dense in K. Since $\mathring{R}_i \cap f^{-1}(\mathring{R}_j)$ is a nonempty open set, we have $Z \cap f^{-1}(Z) \cap \mathring{R}_i \cap f^{-1}(\mathring{R}_j) \neq \emptyset$, so we can take $x \in Z \cap \mathring{R}_i$ with $f(x) \in Z \cap \mathring{R}_j$. Notice that $x \in R(x) \subseteq \mathring{\overline{R(x)}} \Longrightarrow \mathring{R}_i \cap \mathring{\overline{R(x)}} \neq \emptyset \Longrightarrow R_i = \overline{R(x)}$ and, similarly, $R_j = \overline{R(f(x))}$. Using the previous lemma and the fact that g is continuous, we get $g(R_j) = g(\overline{R(f(x))}) \subseteq \overline{g(R(f(x)))} \subseteq \overline{R(x)} = R_i \Longrightarrow R_j = f(g(R_j)) \subseteq f(R_i)$.

Now we will see that there is a semiconjugacy between f and a unilateral subshift of finite type. Let $\{R_1, ..., R_k\}$ be a partition of K as above. We can define a matrix $A \in M_k$ by

$$A_{ij} = 1 \; if \; f(\mathring{R}_i) \cap \mathring{R}_j \neq \emptyset$$
$$A_{ij} = 0 \; otherwise.$$

Lemma 12.5 *Let $(a_0, ..., a_n)$ be an admissible sequence for A. Then, $\bigcap_{i=0}^{n} f^{-i}(\mathring{R}_{a_i}) \neq \emptyset$.*

Proof: The lemma is trivial for sequences with just one element. Suppose now that the lemma is valid for the admissible sequence $(a_1, ..., a_n)$, so that $\bigcap_{i=0}^{n-1} f^{-i}(\mathring{R}_{a_{i+1}}) \neq \emptyset$. Let $y \in \bigcap_{i=0}^{n-1} f^{-i}(\mathring{R}_{a_{i+1}})$. Since $A_{a_0 a_1} = 1$, we have $\mathring{R}_1 \subseteq f(\mathring{R}_0)$. So, $y = f(x)$ for some $x \in \mathring{R}_0$ and it is easy to see that $x \in \bigcap_{i=0}^{n} f^{-i}(\mathring{R}_{a_i})$. $\square$

As a consequence of this lemma, we can see that, for each sequence $\underline{a} = (a_n)_{n \in \mathbb{N}_0} \in \Sigma_A^+$, if $F_n = \bigcap_{i=0}^{n} f^{-i}(R_{a_i})$ then $(F_n)_n$ is a decreasing sequence of nonempty compact sets, so its limit is nonempty. Besides, if x and y are two points in this intersection, then $\forall i \in \mathbb{N}_0, d(f^i(x), f^i(y)) \leq diam(R_{a_i}) < \varepsilon$, so $x = y$. Therefore, we can define a map $\Pi : \Sigma_A^+ \to K$ by

$$\{\Pi(\underline{a})\} = \lim F_n = \bigcap_{n=0}^{\infty} f^{-n}(R_{a_n})$$

Let $\underline{a} \in \Sigma_A^+$. Notice that, since f is surjective, $f(f^{-1}(L)) = L$ for any $L \subseteq K$. Then, we have

$$\{f(\Pi(\underline{a}))\} = f\left(\bigcap_{n=0}^{\infty} f^{-n}(R_{a_n})\right) \subseteq f(R_{a_0}) \cap \bigcap_{n=1}^{\infty} f^{-(n-1)}(R_{a_n}) =$$

$$= f(R_{a_0}) \cap \bigcap_{n=0}^{\infty} f^{-n}(R_{a_{n+1}}) = \bigcap_{n=0}^{\infty} f^{-n}(R_{a_{n+1}}) = \{\Pi(\sigma_A^+(\underline{a}))\}$$

(recall that $A_{a_0 a_1} = 1$ implies $f(R_{a_0}) \supseteq R_{a_1}$). So, $f(\Pi(\underline{a})) = \Pi(\sigma_A^+(\underline{a}))$ and, since Π is surjective and continuous, it is a semiconjugacy of σ_A^+ and f. A point in K can have more than one preimage under Π, but we will show that it can not have more than k preimages.

Lemma 12.6 *Let $(a_0, ..., a_n)$ and $(b_0, ..., b_n)$ be two admissible sequences for A with $a_n = b_n$. If $\forall i \in \{0, \ldots, n\}, R_{a_i} \cap R_{b_i} \neq \emptyset$, then the sequences are equal.*

Proof: We have seen in the previous lemma that $\bigcap_{i=0}^{n} f^{-i}(\mathring{R}_{a_i}) \neq \emptyset$, so there is some $x \in K$ with $f^i(x) \in \mathring{R}_{a_i}$. By hypothesis, $R_{a_n} = R_{b_n}$. Suppose now that, for $i \in [n]$, we have $R_{a_i} = R_{b_i}$. Since $A_{a_{i-1} a_i} = A_{b_{i-1} b_i} = 1$, we get $\mathring{R}_{a_i} \subseteq f(\mathring{R}_{a_{i-1}})$ and $\mathring{R}_{b_i} \subseteq f(\mathring{R}_{b_{i-1}})$. Then, since $f^i(x) \in \mathring{R}_{a_i} = \mathring{R}_{b_i}$ there are $y \in \mathring{R}_{a_{i-1}}$ and $z \in \mathring{R}_{b_{i-1}}$ such that $f^i(x) = f(y) = f(z)$. Also, $d(y, z) \leq diam(R_{a_{i-1}}) + diam(R_{b_{i-1}}) \leq c$ because $R_{a_{i-1}} \cap R_{b_{i-1}} \neq \emptyset$. So, $y = z$ and $\mathring{R}_{a_{i-1}} \cap \mathring{R}_{b_{i-1}} \neq \emptyset$. Since different elements of the partition must have disjoint interior, we conclude that $R_{a_{i-1}} = R_{b_{i-1}}$. $\square$

Therefore,

Proposition 13 *Any point of K has no more than k preimages under Π, where k is the number of rectangles of the partition.*

Proof: Suppose, by contradiction, that there is a point in $x \in K$ with $k + 1$ distinct preimages. Call these preimages $\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^{k+1}$. Then, for n big enough, the admissible sequences $(x_0^i, \ldots, x_n^i)$ must be different from each other. But, since we have $k + 1$ sequences, at least two of them must have the same last element of the sequence, so they should be equal by the previous lemma (recall that, by definition of Π, $f^m(x) \in R_{x_m^i}$ for every $m \in \{0, \ldots, n\}$ and $i \in [k + 1]$). $\square$

Proposition 14 *The preimages of periodic points of f are periodic points of $\sigma = \sigma_A^+$.*

Proof: Suppose that $x \in K$ is such that $f^p(x) = x$ for some $p \in \mathbb{N}$. Let $\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^r$ be the preimages of x, distinct from each other by hypothesis. Then, for every $i \in [r]$, we have $\Pi(\sigma^p(\underline{x}^i)) = f^p(\Pi(\underline{x}^i)) = f^p(x) = x$, so that $\sigma^p(\underline{x}^1), \sigma^p(\underline{x}^2), \ldots, \sigma^p(\underline{x}^r)$ are also preimages of x.

Assume that there are $i, j \in [r]$, $i \neq j$, with $\sigma^p(\underline{x}^i) = \sigma^p(\underline{x}^j)$; in particular, we have $x_p^i = x_p^j$. Then, the admissible sequences $(x_0^i, \ldots, x_p^i)$ and $(x_0^j, \ldots, x_p^j)$ verify the hypothesis of the previous lemma, therefore they must be equal. So, $\underline{x}^i = (x_0^i, x_1^i \ldots, x_p^i, x_{p+1}^i, \ldots) = (x_0^j, x_1^j \ldots, x_p^j, x_{p+1}^j, \ldots) = \underline{x}^j$, which contradicts the assumption that $\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^r$ are distinct from each other.

So, $\sigma^p(\underline{x}^1), \sigma^p(\underline{x}^2), \ldots, \sigma^p(\underline{x}^r)$ are also distinct from each other and, therefore, they are precisely the preimages of x, so that there is some permutation $\mu \in S_r$ such that $\sigma^p(\underline{x}^i) = \underline{x}^{\mu(i)}$ for every $i \in [r]$. So, $\sigma^{ord(\mu)p}(\underline{x}^i) = \underline{x}^{\mu^{ord(\mu)}(i)} = \underline{x}^i$ for every $i \in [r]$. □

Proposition 15 *If $\underline{s}$ and $\underline{t}$ are two preimages of a periodic point $x \in K$ with $s_i = t_i$ for some $i \in \mathbb{N}_0$, then $\underline{s} = \underline{t}$.*

Proof: In fact, since $\underline{s}$ and $\underline{t}$ are both periodic points, there is some common period n such that $\sigma^n(\underline{s}) = \underline{s}$ and $\sigma^n(\underline{t}) = \underline{t}$. Then, the sequences $(s_i, s_{i+1}, \ldots, s_{i+n})$ and $(t_i, t_{i+1}, \ldots, t_{i+n})$ verify the hypothesis of the previous lemma: they end with the same element ($s_{i+n} = s_i = t_i = t_{i+n}$) and, by definition of Π, $f^m(x) \in R_{s_m}$ and $f^m(x) \in R_{t_m}$ for every $m \in \{i, \ldots, i+n\}$. □

For each $r \in [k]$, define:

Definition 15.1 $I_r = \{\{s_1, \ldots, s_r\} \subset [k] : \bigcap_{i=1}^r R_{s_i} \neq \emptyset\}$ *where we assume that $s_1 < s_2 < \ldots < s_r$.*

Definition 15.2 *$A^{(r)}$ and $B^{(r)}$ as matrices with coefficients indexed in the set I_r given, satisfying the following conditions: fixing $s, t \in I_r$ with $s = \{s_1, ..., s_r\}$ and $t = \{t_1, ..., t_r\}$,*

1. *if there is an unique permutation $\mu \in S_r$ such that $A_{s_i t_{\mu(i)}} = 1$ for every $i \in [r]$, then $A_{st}^{(r)} = 1$ and $B_{st}^{(r)} = sgn(\mu)$, where $sgn(\mu)$ denotes the signature of the permutation μ (1 if the permutation is even and -1 if it is odd);*
2. *otherwise, $A_{st}^{(r)} = B_{st}^{(r)} = 0$.*

Let $\Sigma_r^+ = I_r^{\mathbb{N}_0}$ be the set of sequences indexed by $\mathbb{N}_0$ whose elements belong to I_r and $\Sigma(A^{(r)})^+ \subseteq \Sigma_r^+$ be the subset of admissible sequences according to the matrix $A^{(r)}$. Also, let σ_r^+ denote the unilateral shift defined on these sets. Now, we will see how to define a codification map $\hat{\Pi}_r : \Sigma(A^{(r)})^+ \to K$.

Given a sequence $\underline{\hat{a}} = (\hat{a}_n)_n \in \Sigma(A^{(r)})^+$, with $\hat{a}_n = \{a_n^1, ..., a_n^r\} \in I_r$, for every $n \in \mathbb{N}_0$, there is, by definition of $\Sigma(A^{(r)})^+$, an unique permutation μ_n such that $A_{a_n^i a_{n+1}^{\mu_n(i)}} = 1, \forall i \in [r]$. Consider the permutations defined by

$\nu_0 = id$

$\nu_n = \mu_{n-1} \circ \ldots \circ \mu_1 \circ \mu_0$.

Notice that $\mu_n \circ \nu_n = \nu_{n+1}, \forall n \in \mathbb{N}_0$.

For each $i \in [r]$ and $m \in \mathbb{N}_0$, let $\alpha_m^i = a_m^{\nu_m(i)}$. Then, $\underline{\alpha}^i = \left(\alpha_m^i\right)_m$ belongs to Σ_A^+, for every $i \in [r]$. In fact, we have:

$$A_{\alpha_m^i \alpha_{m+1}^i} = A_{a_m^{\nu_m(i)} a_{m+1}^{\nu_{m+1}(i)}} = A_{a_m^{\nu_m(i)} a_{m+1}^{\mu_m(\nu_m(i))}} = 1, \forall m \in \mathbb{N}_0$$

For each $m \in \mathbb{N}_0$, since $\hat{a}_m \in I_r$, we know that there is some $y_m \in \bigcap_{i=1}^r R_{a_m^i}$. So, for all $i, j \in [r]$ we have

$$d(f^m(\Pi(\underline{\alpha}^i)), f^m(\Pi(\underline{\alpha}^j))) \leq d(f^m(\Pi(\underline{\alpha}^i)), y_m) + d(y_m, f^m(\Pi(\underline{\alpha}^j))) \leq$$

$$\leq 2 \max_{n \in [k]} \{diam(R_n)\} < \delta < \varepsilon/2$$

which implies that $\Pi(\underline{\alpha}^i) = \Pi(\underline{\alpha}^j)$.

Hence, for each $r \in [k]$, we can define a map $\hat{\Pi}_r : \Sigma(A^{(r)})^+ \to K$ by setting $\hat{\Pi}_r(\underline{\hat{a}}) = \Pi(\underline{\alpha}^i)$, which does not depend on the choice of the index $i \in [r]$.

Let us verify that $\hat{\Pi}_r(Per_p(\Sigma(A^{(r)})^+)) \subseteq Per_p(f)$. Given $\underline{\hat{a}} \in Per_p(\Sigma(A^{(r)})^+)$, we have

$$\{\hat{\Pi}_r(\underline{\hat{a}})\} = \{\Pi(\underline{\alpha}^i)\} = \bigcap_{n \in \mathbb{N}_0} f^{-n}(R_{\alpha_n^i})$$

for any $i \in [r]$. So,

$$\{\hat{\Pi}_r(\underline{\hat{a}})\} = \bigcap_{i \in [r]} \bigcap_{n \in \mathbb{N}_0} f^{-n}(R_{\alpha_n^i}) = \bigcap_{n \in \mathbb{N}_0} f^{-n}\left(\bigcap_{i \in [r]} R_{\alpha_n^i}\right) = \bigcap_{n \in \mathbb{N}_0} f^{-n}\left(\bigcap_{i \in [r]} R_{a_n^i}\right)$$

and

$$\{f^p(\hat{\Pi}_r(\underline{\hat{a}}))\} = f^p\left(\bigcap_{n \in \mathbb{N}_0} f^{-n}\left(\bigcap_{i \in [r]} R_{a_n^i}\right)\right) \subseteq \bigcap_{n \in \mathbb{N}_0} f^{p-n}\left(\bigcap_{i \in [r]} R_{a_n^i}\right) \subseteq \bigcap_{n \in \mathbb{N}_0, n \geq p} f^{p-n}\left(\bigcap_{i \in [r]} R_{a_n^i}\right) =$$

$$= \bigcap_{n \in \mathbb{N}_0} f^{-n}\left(\bigcap_{i \in [r]} R_{a_{n+p}^i}\right) = \bigcap_{n \in \mathbb{N}_0} f^{-n}\left(\bigcap_{i \in [r]} R_{a_n^i}\right) = \{\hat{\Pi}_r(\underline{\hat{a}})\}$$

because f is surjective and $\hat{a}_n = \hat{a}_{n+p}, \forall n \in \mathbb{N}_0$. So, $f^p(\hat{\Pi}_r(\underline{\hat{a}})) = \hat{\Pi}_r(\underline{\hat{a}})$.

On the other hand, if $x \in Per_p(f)$, let $\underline{\alpha}^1, \ldots, \underline{\alpha}^r$ be the preimages of x under the map Π (notice that $r \leq k$, by a previous proposition). For each $m \in \mathbb{N}_0$ and $i \in [r]$, we have $f^m(x) \in R_{\alpha_m^i}$, so $\bigcap_{i \in [r]} R_{\alpha_m^i} \neq \emptyset$ and, since $\alpha_m^i \neq \alpha_m^j$ for $i \neq j$ (by the previous proposition), we can define an element $\hat{a}_m \in I_r$ and, therefore, build a sequence $\underline{\hat{a}} = (\hat{a}_m)_{m \in \mathbb{N}_0} \in \Sigma_r^+$.

Let us see that $\mu = id$ is the only permutation in S_r such that $A_{\alpha_m^i \alpha_{m+1}^{\mu(i)}} = 1, \forall i \in [r]$. Take a permutation $\mu \in S_r$, with order τ, such that $A_{\alpha_m^i \alpha_{m+1}^{\mu(i)}} = 1, \forall i \in [r]$. Given any $j \in [r]$, consider the two admissible sequences

$$\alpha_n^j \alpha_{n+1}^{\mu(j)} \cdots \alpha_{n+q}^{\mu(j)} \alpha_{n+q+1}^{\mu^2(j)} \cdots \alpha_{n+(\tau-1)q}^{\mu^{\tau-1}(j)} \alpha_{n+(\tau-1)q+1}^j$$

and

$$\alpha_n^j \alpha_{n+1}^j \cdots \alpha_{n+q}^j \alpha_{n+q+1}^j \cdots \alpha_{n+(\tau-1)q+1}^j$$

where q is a common period of the preimages of x. By a previous lemma, they must be equal; in particular, $\alpha_{n+1}^{\mu(j)} = \alpha_{n+1}^j$. Then, the last proposition tells us that $\mu(j) = j$ and, therefore, $\mu = id$.

So, we have $\underline{\hat{a}} \in \Sigma(A^{(r)})^+$. Also, as we have seen before, the set of preimages of x is invariant by σ^p. Then, for each $m \in \mathbb{N}_0$, the element $\hat{a}_{m+p} \in I_r$, whose elements are $\alpha_{m+p}^1, \ldots, \alpha_{m+p}^r$, is the same as the element $\hat{a}_m \in I_r$, because its elements, $\alpha_m^1, \ldots, \alpha_m^r$ are the same (although not necessarily in the same order). Therefore $\hat{a}_{m+p} = \hat{a}_m$, that is, $\underline{\hat{a}} \in Per_p(\Sigma(A^{(r)})^+)$.

The next proposition provides a formula for the number of periodic points of f.

Proposition 16 *For all $p \in \mathbb{N}$,*

$$N_p(f) = \sum_{r=1}^{L} (-1)^{r-1} tr((B^{(r)})^p)$$

where L is the largest value of $r \in [L]$ for which $I_r \neq \emptyset$ (notice that, if $I_r \neq \emptyset$, then $I_{r'} \neq \emptyset$ for $r' < r$).

Proof: Given $x \in Per_p(f)$, consider the function given by

$$\Phi(x) = \sum_{t=1}^{L} \left(\sum_{\underline{\hat{a}} \in \hat{\Pi}_t^{-1}(x) \bigcap Per_p(\Sigma(A^{(t)})^+)} (-1)^{t-1} sgn(\nu) \right)$$

where ν is the unique permutation in S_t such that $\alpha_p^{\nu(i)} = \alpha_0^i, \forall i \in [t]$, with $\underline{\alpha}^i, i \in [t]$ the elements of Σ_A^+ constructed as before. We want to show that $\Phi(x) = 1$. Let $\Pi^{-1}(x) = \{\underline{\alpha}^1, \ldots, \underline{\alpha}^r\}$ and μ be the permutation such that $\sigma^p(\underline{\alpha}^i) = \underline{\alpha}^{\mu(i)}, \forall i \in [r]$, that is, the permutation induced by the action of σ^p on $\Pi^{-1}(x)$. We can write μ as the product of disjoint cycles $\mu_1, \ldots, \mu_s$ (eventually with length 1) which act on the sets $K_1, \ldots, K_s$, respectively, and these sets form a partition of [r].

Given $\underline{\hat{a}} \in \hat{\Pi}_t^{-1}(x)$, we can build t distinct preimages of x under Π, with $t \leq r$. Let $J \subseteq [r]$ be such that these preimages are $(\underline{\alpha}^j)_{j \in J}$. If we suppose additionally that $\underline{\hat{a}} \in Per_p(\Sigma(A^{(t)})^+)$, then J is invariant under ν, so we can write $J = \bigcup_{m \in B} K_m$ for some $\emptyset \neq B \subseteq [s]$. On the other hand, for each nonempty subset B of $[s]$, we can take $J = \bigcup_{m \in B} K_m$ and associate to it a sequence $\underline{\hat{a}}$ given by the set of distinct preimages $(\underline{\alpha}^j)_{j \in J}$.

So, for each $t \in [L]$ and $\underline{\hat{a}} \in \hat{\Pi}_t^{-1}(x) \bigcap Per_p(\Sigma(A^{(t)})^+)$, we can associate an unique nonempty subset B of $[s]$, and we have

$$t = card(J) = card\left(\bigcup_{m \in B} K_m\right) = \sum_{m \in B} card(K_m)$$

Since μ_m is a cycle of length $card(K_m)$, we have

$$sgn(\nu) = \prod_{m \in B} sgn(\mu_m) = \prod_{m \in B} (-1)^{card(K_m)+1} = (-1)^{t+card(B)}$$

Hence,

$$(-1)^{t-1} sgn(\nu) = (-1)^{2t-1+card(B)} = -(-1)^{card(B)}$$

$$\Phi(x) = \sum_{t=1}^{L} \left(\sum_{\underline{\hat{a}} \in \hat{\Pi}_t^{-1}(x) \bigcap Per_p(\Sigma(A^{(t)})^+)} (-1)^{t-1} sgn(\nu) \right) =$$

$$= - \sum_{\emptyset \neq B \subseteq [s]} (-1)^{card(B)} = - \sum_{q=1}^{s} \sum_{B \subseteq [s], card(B)=q} (-1)^{card(B)} =$$

$$= - \sum_{q=1}^{s} \binom{s}{q} (-1)^q = \binom{s}{0} (-1)^0 - \sum_{q=0}^{s} \binom{s}{q} (-1)^q = 1 - (1-1)^s = 1$$

Since $Per_p(\Sigma(A^{(t)})^+) \subseteq \hat{\Pi}_t^{-1}(Per_p(f))$, we have

$$N_p(f) = \sum_{x \in Per_p(f)} \Phi(x) = \sum_{x \in Per_p(f)} \sum_{t=1}^{L} \left(\sum_{\underline{\hat{a}} \in \hat{\Pi}_t^{-1}(x) \bigcap Per_p(\Sigma(A^{(t)})^+)} (-1)^{t-1} sgn(\nu) \right) =$$

$$= \sum_{t=1}^{L} \left(\sum_{\underline{\hat{a}} \in Per_p(\Sigma(A^{(t)})^+)} (-1)^{t-1} sgn(\nu) \right) = \sum_{t=1}^{L} (-1)^{t-1} \left(\sum_{\underline{\hat{a}} \in Per_p(\Sigma(A^{(t)})^+)} sgn(\nu) \right)$$

Let $(\hat{a}_0, ..., \hat{a}_n)$ be an admissible sequence of length $(n+1)$ for the matrix $A^{(t)}$ and let μ_m be the permutation which ensures that $A^{(t)}_{\hat{a}_m \hat{a}_{m+1}} = 1$, for $m \in \{0, 1, ..., n-1\}$. Then, we have $B^{(t)}_{\hat{a}_m \hat{a}_{m+1}} = sgn(\mu_m)$.

Consider the permutations ν_m defined by $\nu_0 = id$ and $\nu_m = \mu_{m-1} \circ ... \circ \mu_0$. We have $\nu_{m+1} = \mu_m \circ \nu_m$ for $m \in \{0, 1, ..., n-1\}$. If $S(\hat{a}_0, \hat{a}_n, n)$ denotes the set of admissible sequences of length $n+1$ which start at $\hat{a}_0$ and end at $\hat{a}_n$, then we can show, by induction over n, that

$$\sum_{S(\hat{a}_0, \hat{a}_n, n)} sgn(\nu_n) = ((B^{(t)})^n)_{\hat{a}_0 \hat{a}_n}$$

For $n = 1$, given two elements $\hat{a}_0, \hat{a}_1 \in I_t$ we have $\nu_1 = \mu_0$, so

$$sgn(\nu_1) = sgn(\mu_0) = (B^{(t)})_{\hat{a}_0 \hat{a}_1}$$

Suppose now this is true for $n = m - 1$. Then, for $n = m$ we have

$$\sum_{S(\hat{a}_0, \hat{a}_m, m)} sgn(\nu_m) = \sum_{S(\hat{a}_0, \hat{a}_m, m)} sgn(\mu_{m-1}) sgn(\nu_{m-1}) =$$

$$= \sum_{\left\{\hat{a}_{m-1} \in I_r : A^{(t)}_{\hat{a}_{m-1} \hat{a}_m} = 1\right\}} \left(\sum_{S(\hat{a}_0, \hat{a}_{m-1}, m-1)} sgn(\nu_{m-1}) \right) sgn(\mu_{m-1}) =$$

$$= \sum_{\left\{\hat{a}_{m-1}\in I_r : A^{(t)}_{\hat{a}_{m-1}\hat{a}_m}=1\right\}} ((B^{(t)})^{m-1})_{\hat{a}_0\hat{a}_{m-1}} B^{(t)}_{\hat{a}_{m-1}\hat{a}_m} = ((B^{(t)})^m)_{\hat{a}_0\hat{a}_m}$$

In particular, we get

$$\sum_{S(\hat{a}_0,\hat{a}_0,n)} sgn(\nu_n) = ((B^{(t)})^n)_{\hat{a}_0\hat{a}_0}$$

As for each sequence $\underline{\hat{a}} \in Per_p(\Sigma(A^{(t)})^+)$ we can associate an unique element of $S(\hat{a}_0,\hat{a}_0,p)$ which verifies $\nu_p = \nu$, we conclude that

$$\sum_{\underline{\hat{a}}\in Per_p(\Sigma(A^{(t)})^+)} sgn(\nu) = \sum_{\hat{a}_0\in I_t} ((B^{(t)})^p)_{\hat{a}_0\hat{a}_0} = \operatorname{tr}((B^{(t)})^p)$$

and therefore

$$N_p(f) = \sum_{t=1}^{L} (-1)^{t-1} \operatorname{tr}((B^{(t)})^p)$$

$\square$

Theorem 16.1 *If f is Ruelle-expanding, then its zeta function is rational.*

Proof: As

$$N_n(f) = \sum_{r=1}^{L} (-1)^{r-1}\operatorname{tr}((B^{(r)})^n) = \sum_{r\in[L], r\, odd} \operatorname{tr}((B^{(r)})^n) - \sum_{r\in[L], r\, even} \operatorname{tr}((B^{(r)})^n)$$

we have

$$\zeta_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n(f)}{n} t^n\right) =$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{\sum_{r\in[L], r\, odd}\operatorname{tr}((B^{(r)})^n) - \sum_{r\in[L], r\, even}\operatorname{tr}((B^{(r)})^n)}{n} t^n\right) =$$

$$= \frac{\exp\left(\sum_{n=1}^{\infty} \frac{\sum_{r\in[L], r\, odd}\operatorname{tr}((B^{(r)})^n)}{n} t^n\right)}{\exp\left(\sum_{n=1}^{\infty} \frac{\sum_{r\in[L], r\, even}\operatorname{tr}((B^{(r)})^n)}{n} t^n\right)} = \frac{\prod_{r\in[L], r\, odd} \exp\left(\sum_{n=1}^{\infty} \frac{\operatorname{tr}((B^{(r)})^n)}{n} t^n\right)}{\prod_{r\in[L], r\, even} \exp\left(\sum_{n=1}^{\infty} \frac{\operatorname{tr}((B^{(r)})^n)}{n} t^n\right)} =$$

$$= \frac{\prod_{r\in[L], r\, odd} \frac{1}{\det(I - tB^{(r)})}}{\prod_{r\in[L], r\, even} \frac{1}{\det(I - tB^{(r)})}} = \frac{\prod_{r\in[L], r\, even} \det(I - tB^{(r)})}{\prod_{r\in[L], r\, odd} \det(I - tB^{(r)})}$$

which is clearly a rational function.

$\square$

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